ELLIPTIC VARIATIONAL INEQUALITY OF THE SECOND KIND FOR THE FLOW OF A VISCOUS, PLASTIC FLUID IN A PIPE.

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Abstract

The elliptic variational inequality of the second kind for the flow of a viscous, plastic fluid in a pipe is considered. This elliptic variational inequality is related to second order partial differential operator. The physical and mathematical interpretation and some properties of the solution are proved.

1- Introduction

The variational inequality is an important and very useful class of non-linear problems arising from mechanics, physics etc. the EVI has two classes, namely EVI of the first kind and EVI of the second kind. In this paper we shall study the existence, uniqueness and properties of the solutions of EVI of the second kind.

1-1: Notations:

- V: real Hilbert space with scalar product (. , .) and associated norm $\|\cdot\|$.
- V*: The dual space of V.
- a(. , .): $V \times V \to \Re$ is a bilinear, continuous and V- elliptic form on $V \times V$.
  A bilinear form a(. , .) is said to be V-elliptic if there exists a positive constant

α such that $a(v,v) \ge \alpha \|v\|^2 \qquad \forall\, v \in V$.

In general we do not assume a(.,.) to be symmetric, since in some applications non-symmetric bilinear forms may occur naturally [1].

- $L: V \to \Re$ continuous , linear functional.
- K : is a closed, convex, non-empty subset of V.
- $j(.): V \to \overline{\Re} = \Re \cup \{\infty\}$ is a convex, lower semi- continuous (l.s.c) and proper functional.

(j(.) is proper if j(v)>-∞ ∀ v∈V and j≠∞).

1.2: EVI of First Kind

To find u∈V such that u is a solution of the problem :

$$P_1 \ldots \begin{cases} a(u, v-u) \ge L(v-u) \quad , \qquad \forall v \in k \\ u \in k \end{cases}$$

1.3: EVI of second kind

To find u∈V such that u is a solution of the problem :

$$P_2 \ldots\ldots \begin{cases} a(u, v-u) + j(v) - j(u) \ge L(v-u) \quad , \forall\ v \in V \\ u \in V \end{cases}$$

1.4: Existence and Uniqueness Results for EVI of Second Kind

1.4.1: A Theorem of Existence and Uniqueness

Theorem 1.4.1 [1]: The problem P_2 has one and only one solution.

Proof:

1- Uniqueness

Let u_1 and u_2 be two solutions of (P_2). Then we have:

$$a(u_1, v-u_1) + j(v) - j(u_1) \geq L(v-u_1) \qquad \forall\, v \in V, u_1 \in V \text{ ....(1)}$$

$$a(u_2, v-u_2) + j(v) - j(u_2) \geq L(v-u_2) \qquad \forall\, v \in V, u_2 \in V \text{ ....(2)}$$

Since j(.) is a proper map there exists $v_o \in V$ such that $-\infty < j(v_o) < \infty$.

Hence for i=1,2

$-\infty < j(u_i) \leq j(v_o) - L(v_o - u_i) + a(u_i, v_o - u_i)$.............................(3)

This shows that $j(u_i)$ is finite for i=1,2. Hence by substituting u_2 for v in (1) and u_1 for v in (2) and adding we obtain

$$\alpha \|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0 \text{ ....(4)}$$

Hence $u_1 = u_2$

2. Existence:

For each $u \in V$ and $\rho > 0$ we associate a problem (Π_ρ^u) of type (P_2) defined as below:-

To find $w \in V$ such that:-

$$(\Pi_\rho^u)\text{....}\begin{cases}(w, v-w) + \rho j(v) - \rho j(w) \geq (u, v-w) + \rho L(v-w) - \rho a(u, v-w) & \forall v \in V \\ w \in V & \text{.....................................(5)}\end{cases}$$

The advantage of considering this problem overt the problem (P_2) is that the bilinear form associated with (Π_ρ^u) is the inner product of V which is symmetric.

Let us first assume that (Π_ρ^u) has a unique solution for all $u \in V$ and $\rho > 0$. For each ρ define the map $f_\rho : V \to V$ by $f_\rho(u) = w$ where w is the unique solution of (Π_ρ^u).

We shall show that f_ρ is a uniformly strict contraction mapping for suitably chosen ρ.

Let u_1, $u_2 \in V$ and $w_i = f_\rho(u_i)$, i=1,2. since j(.) is proper we have $j(u_i)$ finite which can be proved as in (3). Therefore we have

$$(w_1, w_2 - w_1) + \rho j(w_2) - \rho j(w_1) \geq (u_1, w_2 - w_1) + \rho L(w_2 - w_1) - \rho a(u_1, w_2 - w_1), \text{.......(6)}$$

$$(w_2, w_1 - w_2) + \rho j(w_1) - \rho j(w_2) \geq (u_2, w_1 - w_2) + \rho L(w_1 - w_2) - \rho a(u_2, w_1 - w_2), \text{.......(7)}$$

Adding these inequalities we obtain

$$\left.\begin{aligned}\|f_\rho(u_1) - f_\rho(u_2)\|^2 &= \|w_2 - w_1\|^2 \\ &\leq ((I - \rho A)(u_2 - u_1), w_2 - w_1) \\ &\leq \|I - \rho A\| \|u_2 - u_1\| \|w_2 - w_1\|\end{aligned}\right\} \text{..........................(8)}$$

Hence:

$$\|f_\rho(u_1) - f_\rho(u_2)\| \leq \|I - \rho A\| \|u_2 - u_1\|$$

It is easy to show that $\|I - \rho A\| < 1$ when $0 < \rho < \dfrac{2\alpha}{\|A\|^2}$.

This proves that f_ρ is uniformly a strict contracting mapping and hence has a unique fixed point u. This u turns out to be the solution of (p_2) since f_ρ(u)=u implies.

(u, v-u)+ρj(v)- ρj(u)≥(u, v-u)+ ρL(v-u)-ρa(u, v-u) $\forall v\in V$.

Therefore

a(u,v-u)+j(v)-j(u) ≥L(v-u) $\forall v\in V$ ……………………….(9)

Hence (P_2) has a unique solution.

2- An Example of EVI of the second kind

The Flow of A viscous, plastic Fluid in A pipe.

2.1: Notations

* Ω : a bounded domain in $\Re^2$.
* Γ: $\partial\Omega$.
* x={x_1, x_2} a generic point of Ω.
* $\nabla = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}$
* $C^m(\overline{\Omega})$: Space of m-times continuously differentiable real valued functions for which all the derivative up to order m are continuous in $\overline{\Omega}$.
* $C_0^m(\Omega) = \{v \in C^m(\overline{\Omega})$: supp (v) is a compact subset of $\Omega\}$
* $\|v\|_{m,p,\Omega} = \sum_{|\alpha| \le m} \left\| D_v^\alpha \right\|_{L(\Omega)}^p$ for $v \in C^m(\overline{\Omega})$ where α=(α_1, α_2) ; α_1, α_2 non-negative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial X_1^{\alpha_1} \partial X_2^{\alpha_2}}$
* $w^{m,p}(\Omega)$: completion of $C^m(\overline{\Omega})$ in the norm defined above.
* $w_0^{m,p}(\Omega)$: completion of $C_0^m(\Omega)$ in the obove norm.
* $H^m(\Omega) = w^{m,2}(\Omega)$,
* $H_0^m(\Omega) = w_0^{m,2}(\Omega)$.

2.2: The continuous Problem: Existence and Uniqueness results [2].

Let Ω be a bounded domain of $\Re^2$ with a smooth boundary Γ. We define

$$\begin{cases} V = H_0^1(\Omega) \\ a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx \\ L(v) = < f, v > \quad , f \in V^* \\ j(v) = \int_\Omega |\nabla v| dx \end{cases}$$

Let μ, g be two positive parameters, then

Theorem 2.2.1: The variational inequality

$$(10)\ldots\ldots\ldots\ldots \begin{cases} \mu a(u, v-u) + gj(v) - gj(u) \ge L(v-u) \qquad \forall v \in V, \\ u \in V \end{cases}$$

has a unique solution.

Proof:

In order to apply theorem (1.4.1) , we only have to verify that j(.) is convex, proper and l.s.c.

It is obvious that j(.) is convex and proper.

Let u,v $\in$V , then

$\left| j(v) - j(u) \right| \leq \sqrt{meas.(\Omega)} \|u - v\|_V$, ……………………………(11)

hence j(.) is l.s.c.

This proves the theorem.

Remarks

1. If we take g=0 in (10) we recover the variational formulation of the Dirichlet problem

$$\begin{cases} -\mu\Delta u = f & in\,\Omega \\ \text{u=0} & \text{on } \Gamma \end{cases}$$

2. since a(.,.) is symmetric , the solution u of (10) is characterized as the unique solution of the minimization problem

$$(12)\ldots\ldots.. \quad \begin{cases} J(u) \leq J(v) & \forall v \in V \\ u \in V \end{cases}$$

where $J(v) = \dfrac{\mu}{2} a(v,v) + gj(v) - L(v)$

2.3: Physical Motivation

If $L(v) = c\int_\Omega vdx$ (for instance c>0), it is proved in [3] that (10) models the laminar, stationary flow of a Bingham fluid in a cylindrical pipe of cross-section Ω, u(x) being the velocity at x$\in\Omega$[4], [5]. The constant c is the linear decay of pressure and μ, g are respectively. The viscosity and plasticity yield of the fluid. The above medium behaves like a viscous fluid (of viscosity μ) in $\Omega^+ = \{x \in \Omega = |\nabla u(x)| > 0\}$and like a rigid medium in $\Omega^0 = \{x \in \Omega = \nabla u(x) = 0\}$, [6], [7].

2.4: Existence of Multipliers

Let us define A by:

$A = \{q : q \in L^2(\Omega) \times L^2(\Omega), |q(x)| \leq 1 \quad a.e.\}$

$|q(x)| = \sqrt{q_1^2(x) + q_2^2(x)}$, then we have

Theorem 2.4.1

The solution u of (10) is characterized by the existence of p such that

$$(13)\ldots\ldots. \quad \begin{cases} \mu a(u,v) + g\int_\Omega p.\nabla v dx = < f, v > & \forall v \in V, \\ \end{cases}$$

$$
(14)\ldots\ldots\ldots \begin{cases} u \in V \\ p.\nabla u = |\nabla u| \qquad a.e., \\ p \in A \end{cases}
$$

Proof:

We shall prove that (13), (14) imply (10). It follows from (13) that

$$
(15)\ldots\ldots\ldots \begin{cases} \mu a(u, v-u) + g\int_\Omega p.\nabla(v-u)dx = \mu a(u, v-u) + g\int_\Omega p.\nabla v dx - \\ - g\int_\Omega p.\nabla u dx = < f, v-u > \quad \forall v \in V \end{cases}
$$

It follows from (14) that

$$
\int_\Omega p.\nabla x dx = \int_\Omega |\nabla u| dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16)
$$

and from the definition of A that

$$
\int_\Omega p.\nabla u dx \le \int_\Omega |p| \cdot |\nabla v| dx \le \int_\Omega |\nabla v| dx \qquad \forall v \in V \ldots\ldots\ldots\ldots (17)
$$

Then from (13), (15)-(17) we obtain that

$$
\begin{cases} \mu a(u,v-u)+gj(v)-gj(u) \ge < f,v-u > \qquad \forall v \in V, \\ u \in V \end{cases}
$$

Thus (13) and (14) implies (10).

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الخلاصة

تم اعتبار المتباينة التغايرية الناقصية من النوع الثاني لجريان مائع لزج في انبوب. هذه المتباينة التغايرية الناقصية تعود الى مؤثر تفاضلي جزئي من الرتبة الثانية. وقد تم برهنة التفسير الرياضي و الفيزيائي وبعض خواص الحل.